Problem 2. Suppose that a has order 15. Find all left cosets of $\langle a^5 \rangle$ in $\langle a \rangle$.

Solution. We list them in a table. Let $G = \langle a \rangle$ and $H = \langle a^5 \rangle$. Note that a^5 has order three, so we expect $[G : H] = \frac{|G|}{|H|} = \frac{15}{3} = 5$ cosets.

a^4H	a^4, a^9, a^{14}
a^3H	a^3, a^8, a^{13}
a^2H	a^2, a^7, a^{12}
aH	a, a^6, a^{11}
H	$1, a^5, a^{10}$

□

Problem 3. Let a and b be nonidentity elements of different orders in a group G of order 155. Prove that the only subgroup of G that contains a and b is G itself.

Solution. Let $H = \langle a, b \rangle$; it suffices to show that $|H| \geq 155$, from which it would follow that $H = G$. The prime factorization of 155 is $155 = 5 \cdot 31$. Since a and b have different orders, without loss of generality, we may assume that $\text{ord}(a) > \text{ord}(b)$.

If $\text{ord}(a) = 155$, we see that $|H| = 155$, so $H = G$.

Otherwise, $\text{ord}(a) = 31$ and $\text{ord}(b) = 5$. By LaGrange's Theorem, $31 \mid |H|$ and $5 \mid |H|$, so $155 = \text{lcm}(31, 5) \mid |H|$, whence $|H| \geq 155$. □

Problem 4. Let G be a group such that $|G| = pq$, where $p, q \in \mathbb{Z}$ are positive primes. Show that every proper subgroup of G is cyclic.

Solution. Every group of prime order is cyclic. To see this, pick any nonidentity element g from a group of order p, where p is prime. The order of g divides p and is greater than 1, so it must be p. Thus g generates the whole group.

Let H be a proper subgroup of G. We have $|H| < pq$ and $|H| \mid pq$; since p and q are prime, either $|H| = 1$, in which case H is trivial and therefore cyclic (generated by 1), or the order of H is either p or q, so H is cyclic. □

Problem 5. Let $\phi : G \to H$ and $\psi : H \to K$ be group homomorphisms. Show that $\psi \circ \phi : G \to K$ is a group homomorphism.

Solution. Let $g_1, g_2 \in G$. Then

$$\begin{aligned}
(\psi \circ \phi)(g_1 g_2) &= \psi(\phi(g_1 g_2)) && \text{by definition of composition} \\
&= \psi(\phi(g_1)\phi(g_2)) && \text{since } \phi \text{ is a homomorphism} \\
&= \psi(\phi(g_1))\psi(\phi(g_2)) && \text{since } \psi \text{ is a homomorphism} \\
&= (\psi \circ \phi)(g_1)(\psi \circ \phi)(g)2 && \text{by definition of composition.}
\end{aligned}$$

□